QUOTIENTS G/H IN SUPER-SYMMETRY

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Abstract. Main results of Yuta Takahashi and myself [4], together with their background, are reported. Given an affine algebraic super-group scheme G and a closed sub-super-group scheme H over an arbitrary field of characteristic $\neq 2$, we construct the quotient super-scheme G/H, describing explicitly its structure sheaf, and reveals some geometric features of the quotient. One can expect that the results could be applied to investigate representations of a wide class of affine algebraic super-group schemes.

Key Words: super-scheme, super-group scheme, Hopf super-algebra, faisceau.

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1. Background

Throughout we work over a field $\Bbbk$ of characteristic $\neq 2$, and let $\otimes$ denote the tensor product over $\Bbbk$. A $\Bbbk$-*object*, such as algebra, Lie algebra or Hopf algebra (all over $\Bbbk$), is defined on a underlying $\Bbbk$-vector space V, and its structure is given by some linear maps $V^{\otimes n} \to V^{\otimes m}$ between tensor powers. Of course, it is required to satisfy some axioms, that often, as for commutative algebra, or Lie (or Hopf) algebra, involve the symmetry $V \otimes W \xrightarrow{\simeq} W \otimes V$, $v \otimes w \mapsto w \otimes v$. Thus $\Bbbk$-objects are defined based on the symmetric tensor category (or tensor category with symmetry) of vector spaces. The category is generalized to the symmetric tensor category of *super-vector spaces*, based on which defined by the same way of defining the corresponding $\Bbbk$-objects are $\Bbbk$-*super-objects*, such as super-algebra, Lie super-algebra or Hopf super-algebra.

To be more precise the word "super" is a synonym of "graded by the order-2-group $\mathbb{Z}/(2) = \{0, 1\}$". A *super-vector space* is thus a vector space $V = V_0 \oplus V_1$ graded by $\mathbb{Z}/(2)$; a homogeneous element in V_0 (resp., V_1) is said to be *even* (resp., *odd*). The super-vector spaces form a tensor category; the tensor product is the obvious one $V \otimes W$ given the parity $(V \otimes W)_i = \bigoplus_{j=0,1} V_j \otimes W_{i+j}$, $i \in \mathbb{Z}/(2)$, and the unit object is $\Bbbk$; this $\Bbbk$ is supposed to be *purely even*, or namely, consisting of even elements, only. It is indeed symmetric, equipped with the symmetry (or involutive braiding) $V \otimes W \xrightarrow{\simeq} W \otimes V$,

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v = \begin{cases} -w \otimes v & \text{if } v \text{ and } w \text{ are odd,} \\ \ \ w \otimes v & \text{otherwise.} \end{cases}$$

This is called *super-symmetry*.

Vector spaces are precisely purely even super-vector spaces, so that ordinary objects are generalized by super-objects. One might feel this generalization slight. But surprisingly,

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Deligne [2] showed: *Roughly speaking, in characteristic zero, the super-symmetry mostly exhausts all possible symmetries; to be more precise, if $\Bbbk$ is an algebraically closed field of characteristic zero, any rigid, $\Bbbk$-linear abelian symmetric tensor category satisfying a certain mild assumption is realized as the category of finite-dimensional super-modules over some affine super-group scheme* (see the next section for definition), *or in other words, of finite-dimensional super-comodules over some super-commutative Hopf super-algebra.* Motivated by this strong result, I, being a Hopf-algebraist, entered the super-world around 2004.

In what follows (Hopf) super-algebras $A = A_0 \oplus A_1$ are all supposed to be *super-commutative.* To be explicit, this means that A includes A_0 as a central sub-algebra, and we have $ab = -ba$ (and consequently, $a^2 = 0$) for all $a, b \in A_1$.

2. A little of algebraic super-geometry

Let us see quickly how the beautiful framework, founded by A. Grothendieck, of modern algebraic geometry extends to the super context; recall that the framework involves two—geometrical and functorial—view-points, for both of which at base are affine schemes which are equivalent to rings or algebras.

2.1. **From geometrical view-point.** Let A be a super-algebra. A *prime* (resp., *maximal*) *super-ideal* of A may be understood to be a super-ideal presented uniquely as $P \oplus A_1$, where P is a prime (resp., maximal) ideal of A_0. Therefore, A is said to be *local* if A_0 is. A *local super-ringed space* (over $\Bbbk$) is a topological space X which is equipped with a sheaf $\mathcal{O}_X$ of super-algebras (over $\Bbbk$) such that the stalk $\mathcal{O}_{X,P}$ at every point P is local. A *super-scheme* is a local super-ringed space which is locally isomorphic to some affine super-scheme, $\operatorname{Spec} A$. As for this last *affine super-scheme* $\operatorname{Spec} A$, the underlying topological space is the spectrum $\operatorname{Spec}(A_0)$ of A_0 given the Zariski topology, while the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is determined uniquely by

$$\mathcal{O}_{\operatorname{Spec} A}(D(x)) = A \otimes_{A_0} (A_0)_x, \ x \in A_0,$$

where $D(x) = \{P \in \operatorname{Spec}(A_0) \mid x \notin P\}$, and so $\mathcal{O}_{\operatorname{Spec} A, P} = A \otimes_{A_0} (A_0)_P$ for $P \in \operatorname{Spec}(A_0)$.

2.2. **From functorial view-point.** A $\Bbbk$-*functor* is a functor

$$X : (\mathsf{super\text{-}algebras}) \to (\mathsf{sets})$$

from the category of super-algebras (over $\Bbbk$) to the category of sets. A *functorial affine super-scheme* is a representable $\Bbbk$-functor, which is thus of the form

$$\operatorname{Sp} A = \mathsf{SAlg}(A, -) : R \mapsto \mathsf{SAlg}(A, R), \text{ the set of all super-algebra maps } A \to R,$$

where A is a super-algebra. A *functorial super-scheme* is a local $\Bbbk$-functor (see [3, Part I, Section 1.3]) which is the union of some open sub-functors that are functorial affine super-schemes.

2.3. **Comparison and faisceaux.** Almost obviously, $\operatorname{Spec} A \leftrightarrow \operatorname{Sp} A$ gives rises to a category equivalence between the affine super-schemes and the functorial affine super-schemes, which extends, as expected, to a category equivalence between the super-schemes and the functorial super-schemes. The equivalence on one direction associates to every super-scheme X, the $\Bbbk$-functor (indeed, functorial super-scheme)

$$R \mapsto \operatorname{Mor}(\operatorname{Spec} R, X), \text{the set of all morphisms } \operatorname{Spec} R \to X$$

represented by X; see [5, Section 5]. The situation is expressed so as:

$$\begin{array}{ccc}
\text{Geometrical side} & & \text{Functorial side} \\
\begin{pmatrix}\textsf{affine}\\ \textsf{super-schemes}\end{pmatrix} & \xrightarrow{\approx} & \begin{pmatrix}\textsf{functorial affine}\\ \textsf{super-schemes}\end{pmatrix} \\
\cap & & \cap \\
(\textsf{super-schemes}) & \xrightarrow{\approx} & \begin{pmatrix}\textsf{functorial}\\ \textsf{super-schemes}\end{pmatrix} \\
 & & \cap \\
 & & (\textsf{faisceaux})
\end{array}$$

Here, to the functorial side added is the category of faisceaux. A *faisceau* is a $\Bbbk$-functor which behaves like a sheaf for fppf coverings $R \to S$ of super-algebras; an *fppf covering* is a super-algebra map $R \to S$ through which S is fppf (or namely, faithfully flat and finitely presented) over R. Such a covering constitutes the equalizer diagram of super-algebras

$$R \to S \rightrightarrows S \otimes_R S,$$

where the paired arrows indicate $x \mapsto 1 \otimes x,\ x \mapsto x \otimes 1$. A *faisceau* is by definition a $\Bbbk$-functor X such that the induced diagram

$$X(R) \to X(S) \rightrightarrows X(S \otimes_R S)$$

is an equalizer of sets for every fppf covering.

An advantage of the functorial view-point lies in the fact that for every $\Bbbk$-functor X, there exists uniquely a faisceau $\tilde{X}$ together with a natural transformation $X \to \tilde{X}$ which is initial among natural transformations from X to faisceaux. This $\tilde{X}$ is called the *faisceau associated with* X.

2.4. **Affine (algebraic) super-group schemes.** The category of (functorial) super-schemes has finite direct products, so that their group objects are defined, which are called *super-group schemes*. Here we are only interested in affine ones. An affine super-group scheme

$$G = \operatorname{Spec} A = \operatorname{Sp} A$$

corresponds uniquely to a Hopf super-algebra, say A, both from geometrical and functorial view-points. Just as in the ordinary, non-super situation, a left (resp., right) G-super-module is identified with a right (resp., left) A-super-comodule. By saying G-super-modules we will mean left ones unless otherwise stated.

To every super-scheme X, there is naturally associated a scheme X_{ev} with the same underlying topological space $|X|$ as X. For G as above, this G_{ev} is the affine group scheme

$$G_{\mathsf{ev}} = \mathrm{Spec}(A/(A_1)) = \mathrm{Sp}(A/(A_1))$$

that corresponds to the (largest) quotient ordinary Hopf algebra $A/(A_1)$ of A divided by the super-ideal (indeed, Hopf super-ideal) (A_1) generated by A_1. Note that $G = \mathrm{Sp}\, A$ represents a group-valued functor $(\mathsf{super\text{-}algebras}) \to (\mathsf{groups})$, and $G_{\mathsf{ev}} = \mathrm{Sp}(A/(A_1))$ is its restriction $G|_{(\mathsf{algebras})}$ to the category of algebras, or namely, purely even super-algebras. Also, $\mathrm{Spec}(A/(A_1))$ is naturally identified with $\mathrm{Spec}(A_0)$, since $A/(A_1) = A_0/A_1^2$, and A_1^2 consists of nilpotent elements. It follows that the underlying topological spaces of G and of G_{ev} are indeed identified,

$$|G| = |G_{\mathsf{ev}}|. \tag{2.1}$$

We let G be an affine super-group scheme as above, and assume that it is algebraic; this means that the Hopf super-algebra A is finitely generated. Moreover, we let H be a closed sub-super-group scheme of G; this means that H corresponds to a quotient Hopf super-algebra, say B, of A,

$$H = \mathrm{Spec}\, B = \mathrm{Sp}\, B.$$

We concern the following questions. *Does there exist the quotient super-scheme G/H? If yes, does it have desirable properties?* The quotient G/H is defined by the co-equalizer diagram

$$G \times H \rightrightarrows G \to G/H$$

in the category of super-schemes, where the paired arrows indicate the product on $G \times H$ and the projection onto G. In the ordinary, non-super situation it is well known that the quotient exists and it has desirable properties: the quotient scheme G/H is Noetherian, and the quotient morphism $G \to G/H$ is affine and fppf. In the present super situation the G_{ev} associated with G is an affine algebraic group scheme, which includes the H_{ev} associated with H, as a closed sub-group scheme. Therefore the known result can apply to these $G_{\mathsf{ev}} \supset H_{\mathsf{ev}}$.

Let $G \supset H$ be as above. Let $\widetilde{G/H}$ denote the faisceau associated with the $\Bbbk$-functor $R \mapsto G(R)/H(R)$, the set of cosets. Obviously, if $\widetilde{G/H}$ happens to be a super-scheme, then it is the quotient super-scheme. Indeed, Zubkov and I [5] answer the questions in positive, proving that $\widetilde{G/H}$ is a super-scheme. Very recently Takahashi and I [4], on which I am reporting, reproved the result more from the geometrical view-point, by constructing explicitly the structure sheaf of G/H. Some new geometric features of the quotient are revealed, so that our knowledge is now ready to be applied to investigate representations of a wide class of affine algebraic super-group schemes.

3. Main results

Let $G \supset H$ be as in the preceding two paragraphs. There are associated the Lie super-algebras $\mathrm{Lie}(G) \supset \mathrm{Lie}(H)$, on which H, and H_{ev} by restriction, act by adjoint from the right. After restricting the H_{ev}-action to the odd components, we construct first quotient and then dual to obtain the (left) H_{ev}-module

$$\mathsf{Z} = (\mathrm{Lie}(G)_1/\mathrm{Lie}(H)_1)^*.$$

Let $\pi : G_{\mathsf{ev}} \to G_{\mathsf{ev}}/H_{\mathsf{ev}}$ denote the quotient morphism for the associated affine algebraic group schemes. Choose arbitrarily an non-empty affine open sub-scheme U in $G_{\mathsf{ev}}/H_{\mathsf{ev}}$. Then $\pi^{-1}(U)$ is affine, open and right H_{ev}-stable in G_{ev}. By (2.1), $\pi^{-1}(U)$ can be regarded as an open sub-set of $|G|$. Our key result [4, Proposition 4.8] states that there is a non-canonical, right H-equivariant open embedding

$$X_U = \mathrm{Spec}(A_U) \to G \tag{3.1}$$

onto $\pi^{-1}(U)$; therefore, $\pi^{-1}(U)$ is affine, open and right H-stable in G. Here A_U denotes the H-super-module super-algebra given by

$$A_U = (\pi_* \mathcal{O}_{G_{\mathsf{ev}}}(U) \otimes \wedge(\mathsf{Z}) \otimes \mathcal{O}_H(H))^{H_{\mathsf{ev}}}. \tag{3.2}$$

The first two tensor factors $\pi_* \mathcal{O}_{G_{\mathsf{ev}}}(U) = \mathcal{O}_{G_{\mathsf{ev}}}(\pi^{-1}(U))$ and $\wedge(\mathsf{Z})$ on the right-hand side are naturally H_{ev}-modules, while the last $\mathcal{O}_H(H)$ is a left H-, right H_{ev}-super-bimodule; we regard this last as a left H-, left H_{ev}-super-bimodule, by switching the side through the inverse on H_{ev}. The right-hand side of (3.2) indicates the H_{ev}-invariants with respect to the thus obtained left H_{ev}-action on the tensor product. Thus A_U is indeed a left H-super-module super-algebra with respect to the H-action on $\mathcal{O}_H(H)$, and so X_U is right H-equivariant. We can prove that the faisceau $\widetilde{X/H}$, which is constructed just as $\widetilde{G/H}$, is the Noetherian affine super-scheme

$$\mathrm{Spec}\big((\pi_* \mathcal{O}_{G_{\mathsf{ev}}}(U) \otimes \wedge(\mathsf{Z}))^{H_{\mathsf{ev}}}\big),$$

which is isomorphic to

$$Y_U = \mathrm{Spec}\big(\mathcal{O}_G(\pi^{-1}(U))^H\big);$$

the isomorphism is induced from the embedding (3.1). These affine super-schemes have U as the underlying topological space. We can now reproduce our main results, Theorem 4.12, Remark 4.13 and Proposition 4.16, from [4].

Theorem 1. *The Noetherian affine super-schemes Y_U, where U ranges over non-empty affine open sub-sets of $|G_{\mathsf{ev}}/H_{\mathsf{ev}}|$, are uniquely glued into a super-scheme, which is Noetherian, with the underlying topological space $|G_{\mathsf{ev}}/H_{\mathsf{ev}}|$. This super-scheme is the quotient super-scheme G/H of G by H, and represents the faisceau $\widetilde{G/H}$. Moreover, it has the properties:*

- (i) *The quotient morphism $G \to G/H$ is affine and fppf;*
- (ii) *The scheme $(G/H)_{\mathsf{ev}}$ associated with G/H is $G_{\mathsf{ev}}/H_{\mathsf{ev}}$;*
- (iii) *An open sub-set of $|G/H|$ $(= |G_{\mathsf{ev}}/H_{\mathsf{ev}}|)$ is affine as an open sub-super-scheme of G/H, if and only if it is affine as an open sub-scheme of $G_{\mathsf{ev}}/H_{\mathsf{ev}}$;*
- (iv) *The structure sheaf $\mathcal{O}_{G/H}$ is isomorphic to*

  $$\wedge_{\mathcal{O}_{G_{\mathsf{ev}}/H_{\mathsf{ev}}}}\big((\pi_* \mathcal{O}_G \otimes \mathsf{Z})^{H_{\mathsf{ev}}}\big),$$

  restricted on every affine open sub-set. Here $(\pi_* \mathcal{O}_G \otimes \mathsf{Z})^{H_{\mathsf{ev}}}$ is a locally free $\mathcal{O}_{G_{\mathsf{ev}}/H_{\mathsf{ev}}}$-module.

Remark 2. The questions on which we have worked were brought to our interest not long ago, by [1] (2006). In the article Brundan listed up some properties that the quotient G/H should have, and showed some general results, assuming the existence of the quotient. Moreover, he proved that there exists such a quotient G/H with the properties for a special algebraic super-group scheme $G = Q(n)$ and its parabolic sub-super-group schemes $H = P_\gamma$, and applied his general results to $Q(n) \supset P_\gamma$, producing beautiful results on representations of $Q(n)$. Later, Zubkov and I [5] (2011) proved, as was noted already, the existence of quotients in general, showing their properties which, however, do not include one from Brundan's list; see [4, Section 4.4]. The property (iv) above is (a stronger form of) the one which was failed to be shown. We have proved all the properties that Brundan desired and as well, an additional one, the property (iii) above. Thus Brundan's general results are now applicable to affine algebraic super-group schemes in general.

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